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A.G. Miamee, P.I.

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Department of Mathematics
Hampton University
Hampton, Virginia 23668

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13. ABSTRACT (Maximum 200 words)

Stationary stochastic processes have been very useful in analyzing time series appear in applications. However in many engineering application and economic studies there are number of important time series that are not stationary. Hence several authors have been studied non-stationary processes. Several classes of non-stationary processes such as

-Harmonizable processes
- Periodically Correlated (PC) processes
- Almost Periodically Correlated (PAC) processes
- Correlation Autoregressive (CAR) processes

have been introduced and studied..

In this project we have studied the non-stationary processes in general and the PC, APC, and CAR processes in particular obtaining several results which either have been published or submitted for publication in refereed journals.

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A STUDY OF NON-STATIONARY PROCESSES
WITH THEIR APPLICATIONS

FINAL PROGRESS REPORT

by

A.G.Miamee

November 30, 2000

U.S. ARMY RESEARCH OFFICE

GRANT NUMBER DAAH04-96-1-0027

HAMPTON UNIVERSITY
HAMPTON VIRGINIA 23668

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<u>Scientific progress and/or accomplishments</u>: Stationary stochastic processes have been very useful in analyzing time series appear in applications. However in many engineering application and economic studies there are number of important time series that are not stationary. Hence several authors have been studied non-stationary processes. Several classes of non-stationary processes such as

- -Harmonizable processes
- - Periodically Correlated (PC) processes
- - Almost Periodically Correlated (PAC) processes
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have been introduced and studied. The first class was first introduced by Cramer and subsequently studied by several authors. The second and third classes which were first introduced by E.G. Gladyshev have also been studied by many researchers and have been applied to appropriate time series analysis. The last class mentioned which is much richer than the classes of PC and APC processes, was first introduced and studied by J.C. Hardin and this author. An example of this newly introduced CAR processes which is neither PC nor APC is the helicopter noise. Helicopter noise which we hear is the combination of two PC processes generated by the main And rear rotors. It is clear that the sum of two PC processes is not necessarily PC. This idea has been later used by some authors to further study the helicopter noise data.

In this project we have been studying the non-stationary processes in general and the PC, APC, and CAR processes in particular obtaining several results which either have been published or submitted for publication in refereed journals. We intend to continue our study in these directions.

The following papers, where the support of this grant has been acknowledged, have been published or accepted for publication in refereed professional journals. Reprint and/or preprint of these works has been sent to you on an ongoing manner with interim reports.

A. Makagon, C.A. Mensha and A.G. Miamee; A Levinson-type Algorithm for Discrete Stationary Random Fields, SIAM J. Math. Anal. 28 (1997) 897-902

A. Makagon and A.G. Miamee; On the Spectrum of Correlation Autoregressive Sequences, Stochastic Processes and their Applications 69 (1997) 179-193

A. Makagon, A.G. Miamee and B.S.W. Schroeder; Recursive Conditions for Positively of the Angle for Multivariate Stationary Sequences, The Proc. Amer. Math. Soc. 126 (1998) 1821-1825

R. Cheng, A.G. Miamee and M. Pourahmadi; Regularity and Minimality of Infinite Variance Processes, J. Theoret. Probability (to appear)

R. Cheng, A.G. Miamee and M. Pourahmadi; Some External Problems in $L^p(\mu)$, Proc. Amer. Math. Soc. (to appear)

During the latter periods of the grant the following papers, where the support of this grant is acknowledged, have been completed and submitted for publication. copies of preprints of these are enclosed with this report.

On AR(1) model with periodic and almost periodic coefficients (with H. Hurd and A. Makagon

The geometry of $L^p(\mu)$ and growth of moving average coefficients of infinite variance processes (with R. Cheng and M. Pourahmadi)

On Generators of two-parameter semi-groups of operators (with A. Niknam)

On the shift operator for non-stationary processes (with. G.H. Shahkar)

Scientific Personnel: A total of 6 persons have received various types and different levels of support from the funds provided by this grant :
a) Principal Investigator
b) Two research associates
3) Three graduate student.

The Geometry of $L^p(\mu)$ and Growth of Moving Average Coefficients of Infinite Variance Processes

R. Cheng, A.G. Miamee and M. Pourahmadi

Abstract: While the notions of covariance and spectrum are not defined for infinite variance processes, the autoregressive (AR) and moving average (MA) parameters are well defined and it is tempting to characterize prediction-theoretic properties of such processes in terms of these parameters. Attempts are made to determine growth rates for the MA parameters. Some geometric properties of L^p spaces are studied which sheds light on this problem.

1 Introduction

A discrete-time process $\{X_t\}$ with $X_t \epsilon L^p(\Omega, \mathcal{F}, \mathcal{P})$ is said to be *p-stationary* if for all integers $n \geq 1$, $t_1, \cdots, t_n, h$ and scalars $c_1, \cdots, c_n$, $E|\sum_{k=1}^{n} c_k X_{t_k+h}|^p = E|\sum_{k=1}^{n} c_k X_{t_k}|^p$.

Its *innovation process* $\{\epsilon_t\}$ is defined by $\epsilon_t = X_t - P_{H_{t-1}} X_t$ where $P_{H_{t-1}} X_t$ stands for the metric projection of X_t onto $H_{t-1} = \overline{sp}\{X_{t-1}, X_{t-2}, \cdots\}$ in the norm of $L^p(\Omega, \mathcal{F}, \mathcal{P})$. Note that 2-stationary processes as defined above are, indeed, the familiar second-order stationary processes. For $1 < p < 2$, p-stationary processes have no well-defined notions of covariance and spectrum, so that neither the spectral-domain nor the time-domain techniques are as effective as they have been for 2-stationary processes. Recent developments in the prediction theory of stable processes (Cambanis and Soltani, 1984; Cambanis, Hardin and Weron, 1988; Cheng et al., 1998, 2000; Miamee and Pourahmadi, 1988a,b; Makagan and Mandrekar, 1990; Rajput and Sundberg, 1994) have revealed that the two classes of harmonizable and moving average stable processes are disjoint. Cambanis et al. (1988) have discussed additional intriguing prediction-theoretic behavior of discrete time stable processes, unsuspected from the Gaussian or second-order processes.

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It is known (Miamee and Pourahmadi, 1988a) that any nondeterministic p-stationary process can be written as

$$X_t = \epsilon_t + \sum_{k=1}^{n} a_k X_{t-k} + E_{t,n} = \epsilon_t + \sum_{k=1}^{n} b_k \epsilon_{t-k} + V_{t,n}, \tag{1.1}$$

for any $n \geq 1$, where $\{a_k\}$ and $\{b_k\}$ are unique sequences of scalars called the autoregressive (AR) and moving average (MA) parameters of $\{X_t\}$, and $V_{t,n}, E_{t,n} \epsilon H_{t-n-1}$. The second representation in (1.1) is called a finite Wold decomposition of $\{X_t\}$. If the success of characterization of regularity of 2-stationary processes is any clue, then the norm-convergence of $\sum_{k=1}^{n} b_k \epsilon_{t-k}$ as $n \to \infty$, should play a role in the study of regularity of p-stationary processes (Cheng et al., 2000). This question of convergence is, in turn, related to the growth of the MA coefficients $\{b_k\}$. Miamee and Pourahmadi (1988a) have shown that $b_k = O(2^k)$. An improved bound is obtained in the present work for the p-stationary case, using geometric properties specific to $L^p(\mu)$ spaces. Some of the many open problems in this unyielding area of prediction theory are pointed out and appropriate analogues and references are given.

2   The Geometry of $L^p(\mu)$ and its Applications

The notion of James orthogonality in a normed linear space is central to this section. Let x and y be elements of a Banach space $\mathcal{X}$. We write $x \perp_{\mathcal{X}} y$ if $\|x + \alpha y\| \geq \|x\|$ for all scalars α. Note that the relation $\perp_{\mathcal{X}}$ is generally not symmetric or linear.

A Banach space $\mathcal{X}$ is said to be *uniformly convex* if for any $\epsilon \in (0, 2]$ there exists a $\delta_\epsilon > 0$ such that the conditions $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \epsilon$ together imply that

$$\tfrac{1}{2}\|x + y\| \leq 1 - \delta_\epsilon.$$

Here is a useful criterion for uniform convexity.

Proposition 2.1 *A Banach space $\mathcal{X}$ is uniformly convex if and only if the conditions $\|x_n\| \leq 1$, $\|y_n\| \leq 1$ and $\lim_{n \to \infty} \|\frac{1}{2}(x_n + y_n)\| = 1$ together imply that $\lim_{n \to \infty} \|x_n - y_n\| = 0$.*

It is known that for $1 < p < \infty$, the spaces $L^p(\mu)$ are uniformly convex. For the above material, and additional information on Banach spaces see Köthe (1969, p. 353).

Suppose that M is a subspace of a Banach space $\mathcal{X}$. For $x \in \mathcal{X}$ consider the problem of minimizing $\|x - y\|$ over $y \in M$. When $\mathcal{X}$ is uniformly convex, then the extremal vector y is uniquely determined by x and M. In that situation the metric projection mapping $y = P_M x$ is characterized by

$$x - P_M^x \perp M \tag{2.1}$$

If P_M is a metric projection mapping, then

$$\|P_M x\| \le 2\|x\| \tag{2.2}$$

for all $x \in \mathcal{X}$. This is because

$$
\begin{aligned}
\|P_M x\| &= \|P_M x - x + x\| \\
&\le \|x - P_M x\| + \|x\| \\
&\le \|x - P_M x + P_M x\| + \|x\| \\
&= 2\|x\|.
\end{aligned}
$$

We shall see that this bound, derived from general norm properties, can be sharpened when $\mathcal{X} = L^p(\mu)$. Furthermore, from (1.1) and repeated application of (2.2) it follows that

$$|b_m| \le 2^m \cdot \frac{\|X_0\|}{\|\epsilon_0\|} \tag{2.3}$$

for all m. This bound will also be sharpened when using properties special to $L^p(\mu)$ spaces.

Uniform convexity interacts with metric projection in the following way.

Lemma 2.2 *Suppose that the Banach space $\mathcal{X}$ is uniformly convex, M is a subspace of $\mathcal{X}$, and $x \perp_{\mathcal{X}} M$. If $y_m \in M$, and $\lim \|x + y_m\| = \|x\|$, then $\lim \|y_m\| = 0$.*

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Proof. The assertion is trivial if $x = 0$. Otherwise, put $X_m = x/\|x + y_m\|$ and $Y_m = (x + y_m)/\|x + y_m\|$. Note that $\|X_m\| \leq 1$, since $x \perp_\mathcal{X} y_m$, and $\|Y_m\| = 1$. Furthermore,

$$\frac{\|x\|}{\|x + y_m\|} \leq \frac{\|x + \frac{1}{2}y_m\|}{\|x + y_m\|}$$
$$= \|\tfrac{1}{2}(X_m + Y_m)\| \leq 1.$$

By assumption, $\lim \|x\|/\|x + y_m\| = 1$, which then forces $\lim \|\tfrac{1}{2}(X_m + Y_m)\| = 1$. Now Proposition 2.1 gives

$$\lim \|y_m\| = \|x\| \lim(\|y_m\|/\|x\|)$$
$$= \|x\| \lim(\|y_m\|/\|x + y_m\|)$$
$$= \|x\| \lim(\|X_m - Y_m\|) = 0. \quad \square$$

It is known that the metric projection onto a subspace is norm continuous in a strictly convex, locally compact Banach space (Köthe, p. 344.). Here is the result for a uniformly convex space.

Proposition 2.3 *Let M be a subspace of a uniformly convex Banach space $\mathcal{X}$. If $x \in \mathcal{X}$, $x_m \in \mathcal{X}$, and $\lim \|x_m - x\| = 0$, then $\lim \|P_M x_m - P_M x\| = 0$.*

Proof. Observe that

$$\|x - P_M x\| \leq \|x - P_M x_m\|$$
$$\leq \|x - x_m\| + \|x_m - P_M x_m\|$$
$$\leq \|x - x_m\| + \|x_m - P_M x\|$$
$$\leq \|x - x_m\| + \|x_m - x\| + \|x - P_M x\|$$
$$= 2\|x - x_m\| + \|x - P_M x\|.$$

It follows that $\lim \|x - P_M x_m\| = \|x - P_M x\|$. Applying Lemma (2.2), and using the orthogonality condition $(x - P_M x) \perp_\mathcal{X} M$, we get $\lim \|P_M x_m - P_M x\| = 0$. $\quad \square$

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Let $\hat{X}$ be the best predictor of X_0 based on the infinite past

$$\{\ldots, X_{-3}, X_{-2}, X_{-1}\},$$

and $\hat{X}(m)$ be the best predictor of X_0 based on the finite past of length m,

$$\{X_{-m}, \ldots, X_{-3}, X_{-2}, X_{-1}\}.$$

Theorem 2.4 *If $\{X_t\}_{t=-\infty}^{\infty}$ is a p-stationary process, then the finite predictors $\hat{X}(m)$ of X_0 converge in norm to its infinite predictor $\hat{X}$.*

Proof. Let $\{Y_m\}_{m=-\infty}^{\infty}$ be a sequence such that

$$Y_m \in sp\{X_{-m}, \ldots, X_{-3}, X_{-2}, X_{-1}\}$$

and $\lim \|Y_m - \hat{X}\| = 0$; such a sequence exists since

$$\hat{X} \in \overline{sp}\{\ldots, X_{-3}, X_{-2}, X_{-1}\}.$$

With the above definitions we have

$$
\begin{aligned}
\|X_0 - \hat{X}\| &\leq \|X_0 - \hat{X}(m)\| \\
&\leq \|X_0 - Y_m\| \\
&\leq \|X_0 - \hat{X}\| + \|\hat{X} - Y_m\|.
\end{aligned}
$$

From this we see that

$$\lim \|X_0 - \hat{X}(m)\| = \|X_0 - \hat{X}\|.$$

Applying Lemma 2.2, we get

$$\lim \|\hat{X}(m) - \hat{X}\| = 0. \quad \square$$

The following inequalities constitute a parallelogram law for $L^p(\mu)$.

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Proposition 2.5 *If $2 \leq p < \infty$, then for any f and g in $L^p(\mu)$*

$$2(\|f\|^p + \|g\|^p) \quad \leq \quad \|(f+g)\|^p + \|(f-g)\|^p \tag{2.4}$$

$$\leq \quad 2^{p-1}(\|f\|^p + \|g\|^p). \tag{2.5}$$

If $1 < p \leq 2$, then for any f and g in $L^p(\mu)$

$$2^{p-1}(\|f\|^p + \|g\|^p) \quad \leq \quad \|(f+g)\|^p + \|(f-g)\|^p \tag{2.6}$$

$$\leq \quad 2(\|f\|^p + \|g\|^p). \tag{2.7}$$

Equality holds in (2.4) and (2.7), if and only if $fg = 0$ a.e.; equality holds in (2.5) and (2.6) if and only if $f = \pm g$ a.e.

Proof. First, consider the case $p \geq 2$. For any complex numbers a and b, the usual parallelogram law gives

$$|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2). \tag{1}$$

When $p \geq 2$, we have $\| \cdot \|_{l^p} \leq \| \cdot \|_{l^2}$, and so

$$(|a+b|^p + |a-b|^p)^{1/p} \quad \leq \quad (|a+b|^2 + |a-b|^2)^{1/2}$$
$$= \quad 2^{1/2}(|a|^2 + |b|^2)^{1/2}. \tag{2}$$

Apply Hölder's inequality, using

$$\frac{1}{p/2} + \frac{1}{p/(p-2)} = 1$$

to get

$$|a|^2 + |b|^2 \quad \leq \quad (|a|^{2(p/2)} + |b|^{2(p/2)})^{2/p} \cdot (1+1)^{(p-2)/p}$$
$$= \quad (|a|^p + |b|^p)^{2/p} \cdot 2^{(p-2)/p}. \tag{3}$$

Combining this with (2), we find that

$$(|a+b|^p + |a-b|^p)^{1/p} \leq 2^{(p-1)/p}(|a|^p + |b|^p)^{1/p}.$$

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For any f and g in $L^p(\mu)$, we apply the above estimate to $a = f(\omega)$ and $b = g(\omega)$, and integrate to get

$$\|f + g\|^p + \|f - g\|^p \leq 2^{p-1}(\|f\|^p + \|g\|^p) \quad (2 \leq p < \infty). \tag{4}$$

The above argument appears in Köthe (1969, p. 355), in connection with the proof of uniform convexity of $L^p(\mu)$.

For the case $1 < p \leq 2$, let $r = 4/p$ (so that $2 \leq r < 4$) and apply (3) with the parameter $r/2$, yielding

$$|\alpha|^2 + |\beta|^2 \leq (|\alpha|^r + |\beta|^r)^{2/r} \cdot 2^{(r-2)/r}. \tag{5}$$

Taking $|\alpha|^2 = |u|^p$ and $|\beta|^2 = |v|^p$, we get

$$|\alpha|^r = |\alpha|^{2(r/2)}|u|^{p(r/2)} = |u|^{p(2/p)} = |u|^2,$$

and likewise $|\beta|^r = |v|^2$. Hence (4) becomes

$$|u|^p + |v|^p \leq 2^{1-(p/2)} \cdot (|u|^2 + |v|^2)^{p/2}.$$

This is certainly true when $u = a + b$ and $v = a - b$, where a and b are any complex numbers:

$$|a + b|^p + |a - b|^p \leq 2^{1-(p/2)} \cdot (|a + b|^2 + |a - b|^2)^{p/2}. \tag{6}$$

For $1 < p \leq 2$, we have $\|\cdot\|_{l^p} \geq \|\cdot\|_{l^2}$. Applying this fact, along with (4) and (5), brings us to

$$
\begin{aligned}
|a + b|^p + |a - b|^p &\leq 2^{1-(p/2)} \cdot (|a + b|^2 + |a - b|^2)^{p/2} \\
&= 2^{1-(p/2)} \cdot (2|a|^2 + 2|b|^2)^{p/2} \\
&\leq 2 \cdot (|a|^p + |b|^p)
\end{aligned}
$$

As before, for any f and g in $L^p(\mu)$, we apply this estimate to $a = f(\omega)$ and $b = g(\omega)$, and integrate to get

$$\|f + g\|^p + \|f - g\|^p \leq 2(\|f\|^p + \|g\|^p)$$

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The conditions for equality in (2.5)–(2.8) can be traced to the conditions for equality in Hölder's inequality, and the condition $\|\cdot\|_{l^p} = \|\cdot\|_{l^2}$, as these are used in the above arguments.

Note that as p tends to 2 in either direction the Hilbert space case results; the inequalities are sharp in this limited sense.

From the parallelogram law, we get a Pythagorean theorem for $L^p(\mu)$. Again, there are two cases.

Proposition 2.6 *Suppose that $X \perp_p Y$ and define*

$$\lambda = (2^{p-1} - 1)^{-1/p}.$$

If $2 \le p < \infty$, then

$$\|X\|^p + \lambda^p\|Y\|^p \le \|X + Y\|^p. \tag{2.8}$$

If $1 < p \le 2$, then

$$\|X + Y\|^p \le \|X\|^p + \lambda^p\|Y\|^p. \tag{2.9}$$

Proof. We apply (2.4) in the form

$$\left\|\tfrac{1}{2}(f + g)\right\|^p + \left\|\tfrac{1}{2}(f - g)\right\|^p \le \tfrac{1}{2}(\|f\|^p + \|g\|^p). \tag{1}$$

Now taking $f = X$ and $g = X + Y$ in (1) we get

$$\|X + \tfrac{1}{2}Y\|^p + \|\tfrac{1}{2}Y\|^p \le \tfrac{1}{2}\|X\|^p + \tfrac{1}{2}\|X + Y\|^p.$$

Apply (1) repeatedly, taking $f = X$ and $g = X + (1/2^n)Y$, $n = 1, 2, 3, \ldots, N$, will result in

$$2^N\|X + (1/2^{N+1})Y\|^p + 2^N\|(1/2^{N+1})Y\|^p + \cdots + 2^1\|(1/2^{1+1})Y\|^p + 2^0\|(1/2^{0+1})Y\|^p$$

$$\le \quad (2^{N-1} + \cdots + 2^1 + 2^0 + 2^{-1})\|X\|^p + \frac{1}{2}\|X + Y\|^p.$$

Simplifying, taking N to infinity, and using $\|X + (1/2^N)Y\| \ge \|X\|$, we finally get

$$\|X\|^p + \frac{1}{2^{p-1} - 1}\|Y\|^p \le \|X + Y\|^p. \tag{2}$$

Note that the condition $X\perp_p Y$ implies that the quantity $\|X+\alpha Y\|$ is critical when $\alpha=0$. It follows that

$$\lim_{N\to\infty} 2^N(\|X+(1/2^N)Y\|^p - \|X\|^p) = 0,$$

and the estimate leading to (2.8) is asymptotically sharp.

In the case $1 < p \le 2$, we turn to (2.7), with $f=X$ and $g=X+Y$. This yields

$$\tfrac{1}{2}\|X\|^p + \tfrac{1}{2}\|X+Y\|^p \le \|X+\tfrac{1}{2}Y\|^p + \|\tfrac{1}{2}Y\|^p. \tag{3}$$

Now repeat this argument with $f=X$ and $g=X+(1/2^n)Y$, $n=1,2,3,\ldots,N$. The result is

$$(2^N-1)\|X\|^p + \|X+Y\|^p \le 2^N\|X+(1/2^N)Y\|^p + \frac{1}{2^{p-1}-1}\|Y\|^p.$$

Rearranging, we find that

$$\|X+Y\|^p \le \|X\|^p + \frac{1}{2^{p-1}-1}\|Y\|^p + 2^N(\|X+(1/2^N)Y\|^p - \|X\|^p).$$

As N tends to infinity, the last term vanishes, because $X\perp_p Y$. $\quad\square$

Note that equation (2.9) can be sharper than the triangle inequality.

The constant $\lambda = (2^{p-1}-1)^{-1/p}$ appearing in (2.8) and (2.9) might not be optimal, since the estimates in the proof are generally not sharp. One might wonder whether the value $\lambda=1$ is always possible. The following example shows that it is not.

Let $\mathcal{X} = l^3(\{1,2\})$, and consider $f = (1/4,1)$ and $g = (-1,1/16)$ in $\mathcal{X}$. Then $f\perp_3 g$, and

$$\begin{aligned}
\|f\|^3 &= 65/64 \\
\|g\|^3 &= 4097/4096 \\
\|f+g\|^3 &= 6641/4096.
\end{aligned}$$

In order that $\|f\|^3 + \lambda^3\|g\|^3 \le \|f+g\|^3$, it is necessary that $\lambda^3 \le 2481/4097$.

The Pythagorean inequalities give rise to improved bounds on the coefficient growth in the finite Wold decomposition (1.1). As before, we write $\lambda = (2^{p-1}-1)^{-1/p}$.

Theorem 2.7 *Suppose that $\{X_t\}_{t=-\infty}^{\infty}$ is a p-stationary process with nontrivial innovation process $\{\epsilon_t\}_{t=-\infty}^{\infty}$, and finite Wold decomposition (1.2). If $2 \leq p < \infty$, then*

$$\|(1, \lambda b_1, \lambda^2 b_2, \ldots)\|_{l^p} \leq \|X_0\|/\|\epsilon_0\|.$$

Proof. By applying (2.8) repeatedly to the finite Wold decomposition (1.1), we get the bound

$$\|\epsilon_0\|^p + |\lambda b_1|^p \|\epsilon_1\|^p + \cdots + |\lambda^N b_N|^p \|\epsilon_N\|^p + \lambda^N \|V_{0,N}\|^p \leq \|X_0\|^p$$

for all N. Now drop the nonnegative term $\lambda^N \|V_{0,N}\|^p$, and let N increase without bound. $\square$

Observe that this improves on the bound (2.2). The case $1 < p \leq 2$ is more delicate, since the estimate (2.9) is not similarly useful. However, the following can be said.

Proposition 2.8 *Let $1 < p \leq 2$, and suppose that $X \perp_p Y$. If κ is a constant satisfying*

$$0 \leq \kappa \leq (2^{p-1} - 1),$$

then for any positive integer N satisfying

$$N \leq \frac{1}{p-1} \log_2 \left[\frac{\kappa(2^{p-1} - 1) - 1}{2^{p-1} - 2} \right],$$

we have

$$\kappa \|X\|^p + (1 - 2^{-N})\|Y\|^p \leq \|X + Y\|^p. \tag{2.10}$$

Proof. We start with (2.7), using $f = X$ and $g = X + Y$ to get

$$2^{p-1}\|X + \tfrac{1}{2}Y\|^p + 2\|\tfrac{1}{2}Y\|^p \leq \|X + Y\|^p + \|X\|^p.$$

Repeat this estimate using $f = X$ and $g = X + (1/2^n)Y$, $1 \leq n \leq N$, with the result

$$2^{(p-1)N}\|X + (1/2^N)Y\|^p + \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^N} \right)\|Y\|^p$$

$$\leq \quad \|X + Y\|^p + (1 + 2^{p-1} + \cdots + 2^{(p-1)(N-1)})\|X\|^p.$$

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[Unfortunately, the right side generally grows more rapidly than the left, so the argument from the $2 \leq p < \infty$ case is less fruitful.] Rearranging, and using $X \perp_p Y$, we deduce that

$$\left[2^{(p-1)N} - \frac{2^{(p-1)N} - 1}{2^{p-1} - 1}\right] \|X\|^p + (1 - 2^{-N})\|Y\|^p \leq \|X + Y\|^p.$$

The constant enclosed in the square brackets is at most the value $(2^{p-1} - 1)$. For κ satisfying $0 \leq \kappa \leq (2^{p-1} - 1)$, we have

$$\kappa \leq \left[2^{(p-1)N} - \frac{2^{(p-1)N} - 1}{2^{p-1} - 1}\right]$$

whenever

$$N \leq \frac{1}{p-1} \log_2 \left[\frac{\kappa(2^{p-1} - 1) - 1}{2^{p-1} - 2}\right]. \quad \square$$

The values $\kappa = (2^{p-1} - 1)$ and $N = 1$ can always be used, corresponding to the crude bound

$$(2^{p-1} - 1)\|X\|^p + \tfrac{1}{2}\|Y\|^p \leq \|X + Y\|^p. \tag{2.11}$$

The coefficient growth estimate that results from (2.10) is the following.

Corollary 2.9 *Suppose that $\{X_t\}_{t=-\infty}^{\infty}$ is a p-stationary process with nontrivial innovation process $\{\epsilon_t\}_{t=-\infty}^{\infty}$, and finite Wold decomposition (1.2). If $1 < p \leq 2$, then with the notation of Proposition 2.8,*

$$1 + (1 - 2^{-N})|b_1|^p + (1 - 2^{-N})^2|b_2|^p + \cdots \leq \|X_0\|^p / \kappa \|\epsilon_0\|^p.$$

When p is close to 2 (greater than about 1.695), then N is greater than 1, and this is a sharper bound on the coefficient growth than (2.3).

These Pythagorean inequalities also give improved bounds on the norm of the metric projection, compared with the crude result (2.2).

Corollary 2.10 *Let M be a subspace of $L^p(\mu)$. If $2 \leq p < \infty$, then*

$$\|P_M f\| \leq (2^{p-1} - 1)^{1/p}\|f\|.$$

If $1 < p \leq 2$, then

$$\|P_M f\| \leq (1 - 2^{-N})^{-1/p}\|f\|,$$

where N is any positive integer satisfying

$$N \leq \frac{1}{p-1} \log_2 \left[\frac{1}{2 - 2^{p-1}}\right].$$

Again, note that when $1 < p \leq 2$ we can always choose $N = 1$, which gives

$$\|P_M f\| \leq 2^{1/p}\|f\|,$$

still an improvement over (2.2). Furthermore, Corollary 2.10 is sharp in the sense that as p tends to 2 in either direction, we get $\|P_M f\| \leq \|f\|$, which is the correct statement when $p = 2$.

These bounds are generally not sharp. In fact, one might wonder whether $\|P_M x\|$ can actually exceed $\|x\|$. The following example shows that it can. Here, let $\mathcal{X} = l^p(\{1, 2\})$ with $p = 1.1$. Consider $f = (2, 1)$ and $g = (-2, 2^p)$. Then $f \perp_p g$. Take $x = f + g$ and $M = sp\{g\}$. Clearly, $P_M x = g$. We now compute

$$\|x\|^p = (1 + 2^p)^p \approx 3.52\ldots$$
$$\|P_M x\| = 2^p + 2^{(p^2)} \approx 4.45\ldots.$$

For information on the norm of metric projections, see Mazzone and Cuenya (1995).

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On Generators of Two-Parameter Semi-groups of Operators*

A. G. Miamee
Department of Mathematics
Hampton University
Hampton, VA 23664
Abolghassen.miamee@hamptonu.edu

A. Niknam+
Department of Mathematics
Ferdowsi University of Mashad
Mashad P.O. Box 1159-91775, Iran
Niknam @ science2.um.ac.ir

ABSTRACT

Two-parameter semi-group of operators on a Banach space is studied. Generator of such a semi-group is defined in such a way that its domain contain dense subset of analytic elements. Moreover exponential representations for such a semi-group is obtained. These results will extend some well-known results from the theory of one-parameter semi-groups.

1. INTRODUCTION. Let χ be a Banach space and $B(\chi)$ the space of all bounded linear operators on χ. A function

$$t \to U_t$$

from R^+ into $B(\chi)$ is called a one-parameter semi-group of operators if

$$U_{t+s} = U_t U_s, \forall\, t, s \in R^+.$$

A semi-group U_t is called continuous if for every x in χ, the function

$$t \to U_t(x)$$

is continuous at any $t \in R^+$. The infinitesimal generator H of U_t is defined by

$$H(x) = \lim_{\tau \to 0} \frac{U_\tau(x) - x}{\tau}$$

for those x's for which this limit exists.

Domain of H is not χ in general, however it is well-known that this domain is dense in χ. It is even known that the set of analytic elements of U_t, namely the set

$$\mathfrak{R} = \cap_n D(H^n),$$

is dense in χ. Using these facts several exponential representation of U_t in the form

$$U_t = e^{itH} \; ; \; t \in R^+$$

are developed. In this note our purpose is to obtain similar results for the two-parameter semi-groups. By a two-parameter semi-group of operator on χ we mean a function

$$(t,s) \to W_{t,s}$$

from $R_0^+ = R^+ \times R^+ - \{(0,0)\}$ into $B(\chi)$ such that
$$W_{t,s} W_{t',s'} = W_{t+t',s+s'}$$

for every $(t,s), (t',s') \in R_0^+$. Such a two-parameter semi-group $W_{t,s}$ will be called continuous if for each x in χ the function

$$(t,s) \to W_{t,s}(x)$$

is continuous at any point $(t_0, s_0) \in R_0^+$.

To any two-parameter semi-group of operators $W_{t,s}$ we associate two one-parameter semi-groups

$$U_t = W_{t,0} \text{ and } V_t = W_{0,t}, \ t \in R^+$$

If $W_{t,s}$ is continuous so are U_t and V_t. Throughout the rest of this note we assume all semi-groups are continuous, unless otherwise specified. The infinitesimal generators of U_t and V_t are denoted by H and K, respectively. We will think of the pair (H, K) as the generator of $W_{t,s}$ and prove (in Section 2) that its domain, namely $D(H) \cap D(K)$ is dense in χ. We will prove that the set of analytic elements of (H, K), namely

$$\bigcap_{n,m \in N} D(H^n) \cap D(K^m)$$

is dense. Section 3 is devoted to expressing $W_{t,s}$ in exponential form

$$W_{t,s} = e^{itH + isk}, (t,s) \in R_0^+$$

Two parameter semi-groups of operators arise naturally in several areas of applied mathematics including prediction theory of random fields [3 , 4]. Such semi-group of operators can be also used to describe evolution of physical systems in quantum field theory and statistical mechanics [5-7, 9-11].

2. Infinitesimal Generator. In this section, for the sake of completeness, we give a few lemmas stating basic properties of the one parameter semi-groups of operators and their generators.

2.1 Lemma. Let U_t be a semi-group of operators on χ. For any x in χ and any $0 < \alpha < \beta$ denote $x_{\alpha,\beta}$ by

$$x_{\alpha,\beta} = \int_{\alpha}^{\beta} U_\tau(x) dt$$

then $\lim_{n \to \infty} nx_{s,s+\frac{1}{n}} = U_s(x)$, for all $s \in R^+$.

Proof. For any integer n we can write

$$\|nx_{s,s+\frac{1}{n}} - U_s(x)\| = n\left\| \int_s^{s+\frac{1}{n}} \left(U_t(x) - U_s(x)\right)dt\right\| \leq n \int_s^{s+\frac{1}{n}} \|U_t(x) - U_s(x)\| dt.$$

Given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|U_t(x) - U_s(x)\| < \frac{\epsilon}{2}, \quad whenever \quad 1t - s| < \delta.$$

Let N_0 be any integer greater than $\frac{1}{\delta}$. Then for any $n > N_0$ we have

$$\|n\, x_{s,s+\frac{1}{n}} - U_s(x)\| \leq n \int_s^{s+\frac{1}{n}} \|U_t(x) - U_s(x)\| dt$$

$$\leq n \int_s^{s+\frac{1}{n}} \frac{\epsilon}{2} d\,t \leq \frac{\epsilon}{2} < \epsilon.$$

<u>2.2 Lemma.</u> For any semi-group U_t on a Banach space χ and

$$\chi_0 = \bigcup_{t>0} U_t(\chi).$$

(a.) χ_0 Is a linear space

(b.) χ_0 contains all the $x_{\alpha,\beta}$'s   with $x \in \chi$, $0 < \alpha < \beta$.

<u>Proof.</u> (a) is clear because $U_\tau(x)'s$ are nested linear spaces.

(b) For any $x_{\alpha,\beta}$ we can write

$$x_{\alpha,\beta} = \int_\alpha^\beta U_t(x)dt = \int_\alpha^\beta U_{\frac{\alpha}{2}} U_{t-\frac{\alpha}{2}}(x)dt$$

$$= U_{\frac{\alpha}{2}}\left(\int_\alpha^\beta U_{t-\frac{\alpha}{2}}(x)dt\right) = U_{\frac{\alpha}{2}}\left(\int_{\frac{\alpha}{2}}^{\beta-\frac{\alpha}{2}} U_t(x)dt\right)$$

$$U_{\frac{\alpha}{2}}\left(\chi_{\frac{\alpha}{2}\beta-\frac{\alpha}{2}}\right) = U_{\frac{\alpha}{2}}(z) \in U_{\frac{\alpha}{2}}(\chi), \text{ where } z = \chi_{\frac{\alpha}{2}\beta-\frac{\alpha}{2}}$$

So $x_{\alpha,\beta} \in \chi_0$.

2.3 <u>Lemma</u>. With the notation above, we have

$$\overline{sp}\ \{\ x_{\alpha,\beta} : x \in \chi\ ,\ 0 < \alpha < \beta\ \} = \overline{\chi_0}$$

<u>Proof.</u> By Lemma 2.2 we have

$$sp\left\{x_{\alpha,\beta} : x \in \chi, 0 < \alpha < \beta\right\} \subseteq \chi_0.$$

Now take any y *in* χ_0. *Then* $y = U_s(x)$ for some $s > 0$ and some $x \in \chi$.

By Lemma 2.1

$$nx_{s,s + \frac{1}{n}} \to U_s(x) = y \ as \ n \to \infty.$$

This shows that $sp\{x_{\alpha,\beta} : x \in \chi, 0 < \alpha < \beta\}$ is dense in χ_0, which in turn implies that

$$\overline{sp}\{x_{\alpha,\beta} : x \in \chi, 0 < \alpha < \beta\} = \overline{\chi_0}.$$

We can prove more, namely:

2.4 <u>Lemma.</u> With the notation as in lemma 2.3 we have $\overline{sp}\{x_{\alpha,\beta} : x \in \chi, 0 < \alpha < \beta\} = \overline{\chi_0}$.

<u>Proof.</u> It is clear that (by Lemma 2.3)

$$\overline{sp}\{x_{\alpha,\beta} : x \in \chi_0, 0 < \alpha < \beta\} \subseteq \overline{\chi_0}.$$

Now let $y \in \chi_0$ *then* $y = U_s(x)$ for some $x \in \chi_0$ *and some* $s > 0$. We can write y as

$$y = U_s(x) = U_{\frac{s}{2}}U_{\frac{s}{2}}(x) = U_{\frac{s}{2}}(z), \text{ where } z = U_{\frac{s}{2}}(x) \in \chi_0.\ \text{By Lemma 2.1 we can see that}$$

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$$nz_{\frac{s}{2},\frac{s}{2}+\frac{1}{n}} \rightarrow U_{\frac{s}{2}}(z) = y$$

So we found a sequence in

$$sp\{x_{\alpha,\beta} : x \in \chi_0, \ 0 < \alpha < \beta\},$$

namely $nz_{\frac{s}{2},\frac{s}{2}+\frac{1}{n}}$ which converges to y. This means

$$\chi_0 \subseteq \overline{sp}\{x_{\alpha,\beta} : x \in \chi_0, 0 < \alpha < \beta\}$$

and hence

$$\overline{sp}\{x_{\alpha,\beta} : x \in \chi_0, 0 < \alpha < \beta\} = \overline{\chi_0}$$

<u>2.5 Lemma</u>. Let U_t be a semi-group of operator on χ. Then for any x in χ_0

$$\lim_{t \to 0+} U_t(x) = x,$$

<u>Proof.</u> Let x be an element in χ_0, then $x = U_s(z)$ for some s > 0 and some $z \in \chi$.

We can then write

$$\lim_{t \to 0+} U(x) = \lim_{t \to 0+} U_t\left(U_s(z)\right) = \lim_{t \to 0+} U_{t+s}(z)$$
$$= \lim_{\lambda \to s+} U_\lambda(z) = U_s(z) = x.$$

We can improve the last result as follows.

<u>2.6 Lemma</u>. Let U_t be a semi-group of operators on χ. Then

$$\lim_{t \to 0} U_t(x) = x, \ for \ any \ x \in \overline{\chi_0}$$

if and only if $t \to \|U_t \mid \overline{\chi_0}\|$ is bounded in a neighborhood of zero.

<u>Proof.</u> Suppose there exists M > 0 such that $\|U_t \mid \overline{\chi_0}\| \le M$ in a neighborhood, say (0, a), of zero. Let x be an element of $\overline{\chi_0}$. Given $\epsilon > 0$ there exists y in χ_0 such that

$$\| y - x \| < \frac{\epsilon}{2M+1}.$$

By Lemma 2.5

$$\lim_{t \to 0+} U_t (y) = y$$

Hence there exists δ, *with* $0 < \delta < \alpha$, such that

$$\|U_t (y) - y\| < \frac{\epsilon}{2}, \quad whenever \quad |t| < \delta.$$

now for any t with $|t| < \delta$ we can write

$$\|U_t (x) - x\| \le \| U_t (x) - U_t (y)\| + \|U_t (y) - y\| + \|y - x\|$$

$$\le (M+1)\|y - x\| + \|U_t(y) - y\| < (M+1)\frac{\epsilon}{2(M+1)} + \frac{\epsilon}{2} < \epsilon$$

Conversely suppose

$$\lim_{t \to 0+} U_t (x) = x, \quad for \ each \quad x \in \overline{\chi_0}.$$

If we define $U_0 = I$, the identity operator, then for each x *in* $\overline{\chi_0}$, $U_t(x)$ is continuous on [0,1]. So $\|U_t (x)\|$ is continuous and hence bounded on [0, 1] for each x in χ. Applying the uniform boundedness principal [8] to the class $\{U_t : 0 \le t \le 1\}$ of operators on $\overline{\chi_0}$, we conclude that $\{\|U_t|\overline{\chi_0}\|\}$ is bounded on [0, 1].

Using the previous Lemma we can prove:

2.7 <u>Corollary</u>. The set $\{x : \lim_{t \to 0+} U_t(x) = x\}$ Is closed if and only if $\|U_t \mid \chi_0\|$ is bounded in a neighborhood of zero.

2.8 <u>Lemma</u>. We have

$$\{\int_\alpha^\beta U_t(x)dt \ : \ x \in \chi_0\} = \{\int_\alpha^\beta U_t (x)dt \ : \ x \in \chi\}$$

It is clear that the left set is a subset of the right one. To prove the other way let y be in the right

hand side. i.e. let $y = \int_\alpha^\beta U_t(x)dt$ *for some* $x \in \chi$ *and some* $0 < \alpha < \beta$. Then we can write

$$y = \int_\alpha^\beta U_t(x)dt = \int_\alpha^\beta U_{t-\frac{\alpha}{2}}\left(U_{\frac{\alpha}{2}}(x)\right)dt$$

$$= \int_\alpha^\beta U_{t-\frac{\alpha}{2}}(z)\,dt, \; when \; z = U_{\frac{\alpha}{2}}(x) \in \chi_0,$$

so

$$y = \int_\alpha^\beta U_{t-\frac{\alpha}{2}}(z)dt = \int_{\frac{\alpha}{2}}^{\beta-\frac{\alpha}{2}} U_t(z)dz \in L.H.S.$$

The following Lemma explains why we are interested in $x_{\alpha,\beta}$'s

<u>2.9. Lemma.</u> For any $x \in \chi$ and any $0 < \alpha < \beta$ the vector $x_{\alpha,\beta}$ is in the domain of H and

$$H\left(x_{\alpha,\beta}\right) = U_\beta(x) - U_\alpha(x)$$

<u>Proof.</u> Take any vector $x_{\alpha,\beta}$, $x \in \chi$, $0 < \alpha < \beta$. Then we write

$$\lim_{s \to 0} \frac{U_s(x_{\alpha,\beta}) - x_{\alpha,\beta}}{s} = \lim_{s \to 0} \frac{1}{s}\left[\int_\alpha^\beta U_{t+s}(x)dt - \int_\alpha^\beta U_t(x)dt\right]$$

$$= \lim_{s \to 0} \frac{1}{s}\left[\int_{\alpha+s}^{\beta+s} U_t(x)dt - \int_\alpha^\beta U_t(x)dt\right]$$

$$= \lim_{s \to 0} \frac{1}{s}\int_\beta^{\beta+s} U_t(x)dt - \lim_{s \to 0} \frac{1}{s}\int_\alpha^{\alpha+s} U_t(x)dt = U_\beta(x) - U_\alpha(x).$$

Hence $\quad x_{\alpha,\beta} \in D(H)$ and $H\left(x_{\alpha,\beta}\right) = U_\beta(x) - U_\alpha(x)$.

From Lemmas 2.3 and 2.9 we can immediately get the following proposition.

2.10 Proposition. The domain of the infinitesimal generator H of a semi-group U_t of operators on χ is dense in χ_0.

Now we study two parameter semi-groups of operators. Let $W_{t,s}$ be a continuous two-parameter semi-group of operators on χ and consider its two one-parameter semi-groups U_t and V_t of operators on χ, as defined before. Suppose H and K are the infinitesimal generators of H and K, respectively. The following two propositions deals with the domain of infinitesimal generator of $W_{t,s}$, namely $D(H) \cap D(K)$. These results show that H+K has a rich domain.

2.11 Proposition. Let $W_{t,s}$ be a two-parameter semi-group of operators on χ. Then

$$\beta = sp\{\int_\alpha^\beta U_t(x)dt : 0 < \alpha < \beta,\ x \in D(K)\}\ and\ \Re = \{\int_\alpha^\beta V_t(x)dt : 0 < \alpha < \beta, x \in D(H)\}$$

are a subset of $D(H) \cap D(K)$. That is $\Re \cup \beta \subseteq D(H) \cap D(K)$.

Proof. We just show $\beta \subseteq D(H) \cup D(K)$. The other one is similar.
From Lemma 2.9 it is clear that

$$\beta \subseteq D(H).$$

So it remains to show that

$$\beta \subseteq D(K).$$

To see this, pick any x in D(K) then for its corresponding $x_{\alpha,\beta}$ we can write

$$\lim_{s \to 0} \frac{1}{s}\left(V_s\left(x_{\alpha,\beta}\right) - x_{\alpha,\beta}\right) = \lim_{s \to 0} \frac{1}{s}\left(\int_\alpha^\beta \left(V_s\ U_t(x) - U_t(x)\right)dt\right) =$$

$$\lim_{s \to 0}\int_\alpha^\beta U_t\left(\frac{V_s(x)-x}{s}\right)dt = \int_\alpha^\beta U_t\left(\lim_{s \to 0}\left(\frac{V_s(x)-x}{s}\right)dt\right)$$

The last equality follows from the fact that (cf. [2]) $\{\|U_t\|,\ \alpha \le t \le \beta\}$ is bounded for any fixed α, β with $0 < \alpha < \beta$, and the limit in last expression exists, since $x \in D(K)$.

So the $x_{\alpha,\beta}$ belongs to D(K) and $K\left(x_{\alpha,\beta}\right) = \int_{\alpha}^{\beta} U_t\left(K(x)\right)dt.$

From the proof of the previous proposition we can get the following corollary

2.12 Corollary.   With the above notation and for $0 < \alpha < \beta$ we have

(i) $\quad K\left(\int_{\alpha}^{\beta} U_t(x)dt\right) = \int_{\alpha}^{\beta} U_t\left(K(x)\right)dt, x \in D(K),$

(ii) $\quad H\left(\int_{\alpha}^{\beta} V_t(x)dt\right) = \int_{\alpha}^{\beta} V_t\left(H(x)\right)dt, x \in D(H),$

(iii) $\quad H\left(\int_{\alpha}^{\beta} U_t(x)dt\right) = \int_{\alpha}^{\beta} U_t\left(H(x)\right)dt, x \in D(H),$

(iv) $\quad K\left(\int_{\alpha}^{\beta} V_t(x)dt\right) = \int_{\alpha}^{\beta} V_t\left(K(x)\right)dt, x \in D(K).$

2.13 Proposition. Let $W_{t,s}$ be a two-parameter semi-group of operators on χ. With the notation above we have

a) $\quad \mathbb{C} = \{\int_{\alpha}^{\beta}\int_{\alpha}^{\beta} W_{t,s}(x)dtds : x \in \chi\} \leq D(H) \cap D(k)$

b) $\quad \mathbb{C}$ is dense in $\chi_{0,0} = \bigcup_{(t,s) \in R_0^+} W_{t,s}(x)$ and hence $\overline{\chi_{0,0}} = \overline{\mathbb{C}}$.

Proof. (a) Let $y \in \mathbb{C}$ then $y = \int_{\alpha}^{\beta}\int_{\alpha}^{\beta} W_{t,s}(x)dtds$ for some x in χ and some $0 < \alpha < \beta$. We can write

$$\lim_{\tau \to 0+} \frac{1}{\tau}\left[U_\tau\left(\int_{\alpha}^{\beta}\int_{\alpha}^{\beta} W_{t,s}(x)dtds\right) - \int_{\alpha}^{\beta}\int_{\alpha}^{\beta} W_{t,s}(x)dtds\right]$$

$$= \lim_{\tau \to 0+} \frac{1}{\tau}\int_{\alpha}^{\beta}\int_{\alpha}^{\beta}\left(U_{t+\tau}V_s(x) - U_t V_s(x)\right)dtds$$

$$= \int_{\alpha}^{\beta} V_s\left[\lim_{\tau \to 0+} \frac{1}{\tau}\int_{\alpha}^{\beta}\left(U_{t+\tau}(x) - U_t(x)\right)dt\right]ds$$

$$= \int_{\alpha}^{\beta} V_s \left(U_\beta(x) - U_\alpha(x) \right) ds \ ,$$

by lemma 2.9 and the boundedness of $\left\{ \|V_t\| : \prec < t < \theta \right\}$ [2].

This shows that $\mathbb{C} \subseteq D(H)$. Similarly one can show that $\mathbb{C} \subseteq D(K)$.

(b) Let $x \in \chi_{0,0}$. i.e. $x = W_{t,s} \, y$, for some y in χ and some $(t,s) \in R_0^+$. Given any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|t - t'| < \delta \ \text{and} \ |s - s'| < \delta \Rightarrow \|W_{t',s'}(x) - W_{t,s}(y)\| < \frac{\epsilon}{2}$$

Let $N_0 = \left[\dfrac{1}{\delta} \right] + 1$, then for any $n > N_0$ we have

$$\left\| n^2 \int_s^{s+\frac{1}{n}} \int_t^{t+\frac{1}{n}} W_{t',s}(y) dt' ds' - x \right\| =$$

$$\left\| n^2 \int_s^{s+\frac{1}{n}} \int_t^{t+\frac{1}{n}} \left(W_{t',s}(y) - W_{t,s}(y) \right) dt' ds' \right\| \le$$

$$n^2 \int_s^{s+\frac{1}{n}} \int_t^{t+\frac{1}{n}} \| W_{t',s}(y) - W_{t,s}(y) \| dt' ds' \le$$

$$n^2 \int_s^{s+\frac{1}{n}} \int_t^{t+\frac{1}{n}} \frac{\epsilon}{2} \, dt' ds' \le \frac{\epsilon}{2} < \epsilon.$$

The following lemma which is of independent interest can be used to shorten the proof of part (b) in last proposition.

2.14 Lemma. Let $W_{t,s}$ be a two-parameter semi-group of operators on χ. With the notation above, we have

(a) For any x in $\chi_{0,0}$,

$$W_{t,s}(x) \to x, \quad as \quad (t,s) \to (0,0).$$

(b) In order that $\lim\limits_{(t,s)\to(0,0)} W_{t,s}(x) = x$ for all x's in $\overline{\chi_{0,0}}$ it is necessary and sufficient that $\{\|W_{t,s}|\overline{\chi_{0,0}}\|\}$ be bounded in a neighborhood of $(0,0)$.

<u>Proof</u>. Proof is similar to the proofs of Lemmas 2.5 and 2.6

By proposition 2.13 The domain of H + K and (H, K), which is $D(H)\cap D(K)$, is dense in $\overline{\chi_{0,0}}$. We now want to show that (H, K) serves as a generator for $W_{t,s}$, and it possess a dense subspace of analytic elements.

We start with proving the following lemma which will be needed in sequal.

<u>2.15 Lemma</u>. If f is a $C^\infty(0,\infty)$ function with compact support for $n \ge 0$, then

$$\lim_{\tau\to o+} \frac{f^{(n)}(t+\tau) - f^{(n)}(t)}{\tau} = f^{(n+1)}(t), \quad \text{unifo:mly in t.}$$

<u>Proof</u>. Given $\epsilon > 0$ there exists $\delta > 0$, such that

$|f'(s) - f'(t)| < \epsilon$, whenever $|s\text{-}t| < \delta$. Using the mean value theorem we can write

$$\left| \frac{f(t+\tau) - f(t)}{\tau} - f'(t) \right| = |f'(s) - f'(t)|$$

where $t \le s \le t + \tau$. Now if $|\tau| < \delta$ then $|s - t| < \delta$ and hence

$$\left| \frac{f(t+\tau) - f(t)}{\tau} - f'(t) \right| = |f'(s) - f'(t)| < \epsilon,$$

which completes the proof.

<u>2.16 Theorem</u>. Let $W_{t,s}$ be a two parameter semi-group of operators on χ and

U_t, V_t, H and K as before. Let $\Im$ denote the set of all $C^\infty(0, \infty)$ functions with compact support. Then the set

$$g = \left\{ \int_0^\infty \int_0^\infty f(t)g(s)W_{t,s}(x)dtds, x \in \chi, f, g \in \Im \right\}$$

has the following properties

(a) g is a subset of $D(H) \cap D(K)$.
(b) g consists of analytic elements of (H,K).
(c) g is dense in $\overline{\chi}_{0,0}$.

Proof. (a) Let $y \in$ g, then

$$y = \int_0^\infty \int_0^\infty f(t)g(s)W_{t,s}(x)dtds$$

for some $x \in \chi$ and some f, g $\in \Im$. For sufficiently small τ's we can write

$$U_\tau(y) - y = \int_0^\infty \int_0^\infty f(t)g(s)U_\tau W_{t,s}(x)dtds - \int_0^\infty \int_0^\infty f(t)g(s)W_{t,s}(x)dtds$$

$$= \int_0^\infty \int_0^\infty f(t)g(s)W_{t+\tau,s}(x)dtds - \int_0^\infty \int_0^\infty f(t)g(s)W_{t,s}(x)dtds$$

$$= \int_0^\infty \int_0^\infty f(t-\tau)g(s)W_{t,s}(x)dtds - \int_0^\infty \int_0^\infty f(t)g(s)W_{t,s}(x)dtds$$

$$= \int_0^\infty \int_0^\infty (f(t-\tau) - f(t))(x)g(s)W_{t,s}(x)dtds.$$

So we can write

$$\lim_{\tau \to 0+} \frac{U_\tau(y) - y}{\tau} = \lim_{\tau \to 0+} \int_0^\infty \int_0^\infty \frac{f(t-\tau) - f(t)}{\tau} g(s)W_{t,s}(x)dtds$$

We claim this limit exists and is equal to

$$-\int_0^\infty \int_0^\infty f'(t)g(s)W_{t,s}(x)\,dt\,ds.$$

Since f and g are in $\Im$ there exists α,β with $0 < \alpha < \beta$ such that f and g are zero outside $[\alpha,\beta]$. Since $\|g(s)W_{t,s}(x)\|$ is continuous on $[\alpha,\beta] \times [\alpha,\beta]$ it in bounded there. i.e. there exists some $M > 0$ such that

$$\|g(s)W_{t,s}(x)\| \leq M, \text{ for all } t,\tau \in [\alpha,\beta].$$

Given $\in > 0$, by Lemma 2.15, there exists a positive $\delta < \dfrac{\alpha}{2}$ such that

$$|\tau| < \delta \Rightarrow \left| \frac{f(t-\tau)-f(t)}{\tau} - f'(t) \right| < \frac{\in}{2M\beta^2}$$

For $|\tau| < \delta$ we can write

$$\left\| \int_0^\infty \int_0^\infty \frac{f(t-\tau)-f(t)}{\tau} g(s)W_{t,s}(x)\,dt\,ds + \int_0^\infty \int_0^\infty f'(t)g(s)W_{t,s}(x)\,dt\,ds \right\|$$

$$\leq \left\| \int_0^\infty \int_0^\infty \left(\frac{f(t-\tau)-f(t)}{\tau} + f'(t) \right) g(s)W_{t,s}(x)\,dt\,ds \right\|$$

$$\leq \int_0^\infty \int_0^\infty \left| \frac{f(t-\tau)-f(t)}{\tau} + f'(t) \right| \|g(s)W_{t,s}(x)\|\,dt\,ds$$

$$\leq \int_\alpha^\beta \int_\alpha^\beta \left(\frac{\in}{2M\beta^2} \right)(M)\,dt\,ds = \frac{M\in}{2M\beta^2}(\beta-\alpha)^2 \leq \frac{\in}{2}$$

This proves our claim that y is in D (H) and

$$H(y) = -\int_0^\infty \int_0^\infty f'(t)g(s)W_{t,s}(x)dtds.$$

Similarly one can show that y is in D(K) and

$$K(y) = \int_0^\infty \int_0^\infty -f(t)g'(s)d\, W_{t,s}(x)dtds.$$

(b) similar to part (a) one can show that $y = \int_0^\infty \int_0^\infty f(t)g(s)W_{t,s}(x)dtds$ is in

$D(H^m K^n)$, for every m, n>,0.

Furthermore, we have

$$(H^m K^n)(y) = (-1)^{m+n} \int_0^\infty \int_0^\infty f^{(m)}(t)g^{(n)}(s)W_{t,s}(y)dtds$$

So any element in g is an analytic element.

Note that the analytic elements of W$_{t,s}$ are those in $\bigcap_{m,n\geq 0} D(H^m K^n)$

(c) suppose $x*$ is a bounded linear functional on χ which vanished on g . i.e. suppose

$$x^*\left(\int_0^\infty \int_0^\infty f(t)g(s)W_{t,s}(x)dtds \right) = 0$$

for every x in χ and every $f, g \in \Im$. We can therefore write

$$\int_0^\infty \int_0^\infty f(t)g(s)x^*\left(W_{t,s}(x)\right)dtds = 0,$$

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for every $f,g \in \Im$ and every x in χ. This, we claim, implies

$$x^*\left(W_{t,s}(x)\right) = 0, \text{ for all } x \in \chi, (t,s) \in R_0^+,$$

which means x^* vanishes on $\chi_{0,0.}$

This, in virtue of the Hahn-Banach Theorem [8] shows that g is dense in $\overline{\chi_{0,0}}$. So to complete the proof we need to check our claim above that

$$x^*\left(W_{t,s}(x)\right) = 0, \text{ for all } x \in \chi, (t,s) \in R_0^+.$$

Suppose on the contrary

$$x^*\left(W_{t_0,s_0}(x_0)\right) \neq 0,$$

for some $x_0 \in \chi$ and some $(t_0, s_0) \in R_0^+$. WLOG we can assume that the real part of $x^*\left(W_{t_0 \cdot s_0}(x_0)\right)$ is positive, i.e.

$$\text{Re } x^*\left(w_{t_0,s_0}(x)\right) > 0.$$

Since $\chi^*\left(W_{t,s}(x_0)\right)$ and hence $\mathbf{Re}\left(x^*\left(W_{t,s}(x_0)\right)\right)$ is continuous in (t, s), there exists a neighborhood $(\alpha,\beta) \times (\gamma,\delta)$ of (t_0,s_0) Such that

$$\text{Re}\left(W_{t,s}(\chi_0)\right) > 0, \forall (t,s) \in (\alpha,\beta) \times (\gamma,\delta).$$

Now take any two nonzero functions f and g in $\Im$. By a linear change of variable we can find two other such functions, we call them f and g again, which are zero out side (α,β) and (γ,δ), respectively. Squaring these functions if necessary, we can further assume that f and g are positive throughout (α,β) and (γ,δ), respectively. For these functions f and g we have

$$\int_0^\infty \int_0^\infty f(t)g(s)W_{t,s}(x_0)dtds = \int_\alpha^\beta \int_\gamma^\delta f(t)g(s)W_{t,s}(x_0)dtds > 0$$

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which is a contradiction.

3. EXPONENTIAL REPRESENTATION.. In this section we would extend the one-parameter semi-group results concerning their exponential representation. Let $W_{t,s}$ be a two-parameter semi-group of operators on χ with U_t and V_t being its associated one-parameter semi-groups as defined before. For each $\tau > 0$ we define two bounded operators as follows

$$H_\tau(x) = \frac{U_\tau(x) - x}{\tau}, and\ K_\tau(x) = \frac{V_\tau(x) - x}{\tau}$$

The infinitesimal generators H and K of U_t and V_t, namely

$$H(x) = \lim_{\tau \to 0} H_\tau(x) \text{ and } K(x) = \lim_{\tau \to 0} K_\tau(x)$$

are densely defined. For any bounded operator A, the operator e^A is defined by

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

It is clear that the series on the right hand side is absolutely convergent and one has

$$\left\| e^A \right\| \le e^{\|A\|}.$$

If the operator A is not bounded k and / or not everywhere defined then by e^A we would mean the operator defined by

$$\left(e^A\right)(x) = \sum_{k=0}^{\infty} \frac{A^k(x)}{k1},$$

which is defined on those x in $\bigcap_{n \ge 0} D\left(A^n\right)$ for which the series is summable.

There are several exponential representations for a one-parameter semi-group of operator (cf. [1,2]).

Here is one which we want to extend to the two-parameter case.

<u>3.1 Theorem.</u> Let $U_t, t \in R^+$ be any one-parameter semi-group of operator on χ with

$$H_\tau(x) = \frac{U_\tau(x) - x}{\tau}.$$ For every $x \in \chi$ we have

$$U_t(x) = \lim_{\tau \to 0+}\left(e^{itH_\tau}\right)(x),$$

where the convergence is uniform with respect to t in compact sets.

<u>3.2 Theorem</u> (Exponential representation) Let $W_{t,s}$ be a two-parameter semi-group of operators on the Banach space χ and let $\dot{U}_t, V_t, H_\tau$ and K_τ be as before. Then

$$W_{t,s} = \lim_{\tau \to 0} e^{itH_z + isK_z}$$

where convergence is in the strong sense and uniformly on compact sets.

<u>Proof.</u> Let x be a fixed vector in χ and C be a compact subset of R^+. By Theorem 3.1

$$\left(e^{itH_\tau}\right)(x) \to U_t(x)$$

As $\tau \to 0$, uniformly for all t in C. So there exists a $\delta_1 > 0$ such that

$$t \in C \text{ and } 0 < \tau < \delta_1 \Rightarrow \left\|e^{itH_\tau}(x) - U_t(x)\right\| < 1$$
$$\Rightarrow \left\|e^{itH_\tau}(x)\right\| \le \left\|U_t(x)\right\| + 1$$

on the other hand by [8] there exist M >0 such that

$$\left\|U_t(x)\right\| \le M \text{ and } \left\|V_t(x)\right\| \le M$$

for all $t \in C$. Thus $t \in C$ and $0 < \tau < \delta_1 \Rightarrow \left\|e^{itH_\tau}(x)\right\| \le M + 1$

Applying uniform boundedness principal to the class

$$\left\{e^{itH_\tau}\right\}_{t \in c, 0 < \tau < \delta_1}$$

we conclude that there exists some L>0 such that

$$\left\| e^{itH_\tau} \right\| < L$$

for all $t \in C$ and $\tau \in (0, \delta_1)$.

Let $x \in \chi$ and C a compact subset of R^+ be fixed. Given $\epsilon > 0$, applying Theorem 3.1 to U_t *and* V_t there exists positive number δ_2 and δ_3 such that

$$t \in C \text{ and } 0 < \tau < \delta_2 \Rightarrow \left\| e^{itH_\tau}(x) - U_t(x) \right\| < \frac{\epsilon}{2M}$$

$$s \in C \text{ and } 0 < \tau < \delta_3 \Rightarrow \left\| e^{itH_\tau}(x) - U_t(x) \right\| < \frac{\epsilon}{2L}$$

Letting $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, for any $t, s \in C$ and $0 < \tau < \delta$, we have

$$\left\| e^{itH_\tau + isK_\tau}(x) - W_{t,s}(x) \right\| =$$

$$\left\| e^{itH_\tau + isK_\tau}(x) - U_t V_s(x) \right\| \leq$$

$$\left\| e^{itH_\tau}\left(e^{isK_\tau}(x) - V_s(x) \right) \right\| + \left\| \left(e^{itH_\tau} - U_t \right) V_s(x) \right\|$$

$$\leq L \left\| e^{isK_\tau}(x) - V_s(x) \right\| + \left\| e^{itH_\tau} - U_t \right\| M$$

$$\leq L \frac{\epsilon}{2L} + \frac{\epsilon}{2M} M \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon.$$

<u>3.3 Remark.</u>   In all the results above one can replace the original Banach space χ by $\overline{\chi}_0$ *or* $\overline{\chi}_{0,0}$, accordingly. $\chi_0 = U_{t>0} U_t(\chi)$ and $\chi_{0,0} = U_{(t,s) \in R_0^+} W_{t,s}(\chi)$ are actually the spaces on which our semi-groups live. Considering this we can then forget about χ_0 in all these results and change their statements with $\overline{\chi}_0$ replaced by χ .

<u>3.4 Remark</u>. Here, in this note we said a semi-group $t \to U_t$ to be continuous if

$$\lim_{t \to s} U_t(x) = U_s(x)$$

for every x in χ and every $s > 0$. . Some other authors say $t \to U_t$ is continuous if it is continuous strongly at each $s \geq 0$. i.e. if

$$\lim_{t \to s} U_t(x) = \begin{cases} U_s(x), & \text{if } s > 0 \\ x & , \text{ if } s = 0 \end{cases}$$

In this case defining $U_0 = I$, continuity will mean

$$\lim_{t \to s} U_t(x) = U_s(x), \text{ for all } s \geq 0.$$

and $\overline{\chi_0}$ becomes all of χ itself. To see this one only need to check $\chi \subseteq \overline{\chi_0}$.

For, checking this let $x \in \chi$ Then we can write

$$x = \lim_{t \to 0+} U_t(x)$$

which means $x \in \overline{\chi}_0$.

<u>3.5 Remarks</u>. One can study n-parameter semi-groups of operators for $n > 2$ and obtain similar result.

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